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EMBEDDING FOUR MANIFOLDS IN $\mathbf{R}^7$

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HAEFLIGER and Hirsch proved in [8] that a closed smooth n-manifold ($n \neq 4$) embeds smoothly in $\mathbf{R}^{2n-1}$ if and only if the normal Stiefel–Whitney class $\bar{w}_{n-1}(M) = 0$. A PL case analogue of dimension 4 of their Theorem was proved in [9] by Hirsch. It is natural to ask if the above Theorem is also valid in dimension 4. The smooth embedding of orientable four manifolds were considered by several authors [2], [16]. In [2] Boéchat and Haefliger proved that a closed oriented smooth four manifold M embeds smoothly in $\mathbf{R}^7$ if and only if there exists a class $\omega \in H^2(M; \mathbf{Z})/_{\text{torsion}}$ satisfying $\omega^2 = \operatorname{sign}(M)$ and $\omega x \equiv x^2 (\operatorname{mod} 2)$ for every $x \in H^2(M; \mathbf{Z})/_{\text{torsion}}$. In this paper we will generalize (and reprove) their result to nonorientable four manifolds. We also investigate the question when a topological four manifold embeds locally flat in $\mathbf{R}^7$. As a corollary of a Theorem of Donaldson [6] the answer for orientable smooth four manifolds simplifies.

MAIN THEOREM (A). *A smooth closed four manifold embeds smoothly into $\mathbf{R}^7$ if and only if $\bar{w}_3(M) = 0$.*

(B). *A topological closed four manifold embeds locally flat into $\mathbf{R}^7$ if and only if*

(i) *In the nonorientable case, $\bar{w}_3(M) = 0$ and $KS(M) = 0$.*

(ii) *In the orientable case, there exists a class $\omega \in H = H^2(M; \mathbf{Z})/_{\text{torsion}}$ so that*

$$\omega^2 = \operatorname{sign}(M), \omega x \equiv x^2 (\operatorname{mod} 2) \quad \textit{for every } x \in H, \textit{ and } KS(M) = 0,$$

where $KS(M) \in \mathbf{Z}_2$ is the Kirby–Siebenmann obstruction.

An interesting byproduct of our method is

COROLLARY 1. *If M is a closed nonorientable smooth (topological) four manifold with $\bar{w}_3(M) = 0$ (and $KS(M) = 0$), then there is a smooth (topological) immersion $f: M \to \mathbf{R}^7$ which cannot be regularly homotopic to an embedding.*

By Freedman [7] every symmetric unimodular form over integers is realized as the intersection form of a closed topological four manifold with trivial Kirby–Siebenmann obstruction. The Theorem (B) (ii) leads to an interesting problem on quadratic forms. Using information about modular forms one can obtain

COROLLARY 2. *Let M be an oriented closed topological four manifold with $KS(M) = 0$, then M embeds into $\mathbf{R}^7$ locally flat if one of the following conditions is fulfilled:*

(i) *$\mathbf{I}_M$ is indefinite or rank $H_2(M; \mathbf{Q}) \leqslant 16$,*

(ii) *$\mathbf{I}_M$ is positive definite even type and rank $H_2(M; \mathbf{Q}) \leqslant 72$,*

where $\mathbf{I}_M$ is the intersection form of M.

In the proof we will in sections 1 and 2 consider the smooth case. In section 3 we discuss the necessary changes for the topological case and the proof of the second corollary.

1. THE SPIN BORDISM GROUP $\Omega_6^{\mathrm{SPIN}}(K(G;2))$

In order to prove our Theorem it is necessary to analyze the spin bordism group $\Omega_6^{\mathrm{spin}}(K(\mathbf{Z};2))$ and $\Omega_6^{\mathrm{spin}}(K(\mathbf{Z}_2;2))$, where $K(\mathbf{G};2)$ is the Eilenberg–Maclane space.

PROPOSITION 1.1. *$\Omega_6^{\mathrm{spin}}(K(\mathbf{Z};2)) \cong \mathbf{Z} \oplus \mathbf{Z}$ and a spin bordism class $[M,s,f] \in \Omega_6^{\mathrm{spin}}(K(\mathbf{Z};2))$ is zero if and only if*

(1). $\langle f^*(x^3), [M] \rangle = 0$

(2). $\langle p_1(M) \cup f^*(x), [M] \rangle = 0$

where $x \in H^2(K(\mathbf{Z};2))$ is a generator and $p_1(M)$ the first Pontrjagin class.

Proof. First note the odd torsion of $\Omega_6^{\mathrm{spin}}(K(\mathbf{Z};2))$ is isomorphic to that of $\Omega_6^{\mathrm{so}}(K(\mathbf{Z};2))$ which is zero by Conner–Floyd [4] Theorem 17.5. Now by Stong [15] p. 354 it follows that $\Omega_6^{\mathrm{spin}}(K(\mathbf{Z};2)) \cong \Omega_4^{\mathrm{spin}^c}$. The Thom spectrum $M\mathrm{Spin}^c$ has the following well known 2-primary decomposition

$$M\mathrm{Spin}^c(2n) \simeq \prod_I B\mathrm{U}(2n + 4n(I), \ldots\ldots) \times \prod_i K(\mathbf{Z}_2; 2n + \deg z_i)$$

for n large and our range it is $B\mathrm{U}(2n, \ldots\ldots) \times B\mathrm{U}(2n+4, \ldots\ldots) \times \ldots\ldots$ Hence

$$\Omega_6^{\mathrm{spin}}(K(\mathbf{Z};2)) \cong \Omega_4^{\mathrm{spin}^c} \cong \mathbf{Z} \oplus \mathbf{Z}$$

Since the forgetful homomorphism

$$\Omega_6^{\mathrm{spin}}(K(\mathbf{Z};2)) \to \Omega_6^{\mathrm{so}}(K(\mathbf{Z};2))$$

is a rational isomorphism, thereby it is also a monomorphism. The others of the Proposition follow by [4] Theorem 17.5.

PROPOSITION 1.2. *$\Omega_6^{\mathrm{spin}}(K(\mathbf{Z}_2;2)) \cong \mathbf{Z}_2$ and the homomorphism*

$$\Omega_6^{\mathrm{spin}}(K(\mathbf{Z}_2;2)) \to \mathbf{Z}_2$$

$$[M,s,f] \mapsto \langle f^*(x^3), [M] \rangle \in \mathbf{Z}_2$$

is an isomorphism, where $x \in H^2(K(\mathbf{Z}_2;2), \mathbf{Z}_2)$ is the generator.

Proof. First we claim that $\Omega_5^{\mathrm{spin}}(K(\mathbf{Z}_2;2)) = 0$. Let $[M,s,f] \in \Omega_5^{\mathrm{spin}}(K(\mathbf{Z}_2;2))$, if $\alpha \in \pi_1(M)$, up to homotopy we can choose an embedding $\alpha\colon S^1 \times D^4 \to M$ agree with the spin structure s and represents the homotopy class α. By homotopy extension Theorem it can be assumed that $f|_{\alpha(S^1 \times D^4)}$ is constant. Surger $[M,s,f]$ by using α we obtain a triple $[M',s',f']$ and the bordism class is unchanged, hence we can assume M is 1-connected. Similarly, if $\alpha \in H_2(M), f_*(\alpha) = 0$, represents α by an embedded handle and surger M by it, then we get another representation of the bordism class $[M,s,f]$. By Smale [14] the torsion of $H_2(M)$ is isomorphic to $\mathbf{T} \oplus \mathbf{T}$, where $\mathbf{T}$ is a torsion group. Hence the torsion of $\operatorname{Ker} f_*$ is also nonzero if tor $H_2(M)$ is nonzero. If $\alpha \in \operatorname{Ker} f_*$ is a finite order nonzero element, surger by α we get a triple $[M',s',f']$ with the order of torsion $H_2(M')$ less than the order of torsion $H_2(M)$ and rank $H_2(M') = $ rank $H_2(M) + 1$. Continuing this process we can replace $[M, s, f]$ by a triple with M simply connected and $H_2(M)$ torsion free. Now surger M again by some primitive elements of $H_2(M)$, we can assume $H_2(M) \cong \mathbf{Z}$. Hence M is diffeomorphic to $S^2 \times S^3$ (c.f: Small [14]). Note $S^2 \times S^3$ admits an unique spin structure and f_0: $S^2 \times S^3 \to K(\mathbf{Z}_2; 2)$ can extend to a f_0': $S^2 \times D^4 \to K(\mathbf{Z}_2; 2)$. Hence $[S^2 \times S^3, s, f_0] = 0$. Thus $[M, s, f] = 0$ and $\Omega_5^{\text{spin}}(K(\mathbf{Z}_2; 2)) = 0$.

Note $\Omega_6^{\text{spin}}(K(\mathbf{Z}_2; 2))$ has no odd torsion, by Anderson–Brown–Peterson [1] we have the isomorphism

$$\Omega_6^{\text{spin}}(K(\mathbf{Z}_2; 2)) \cong \pi_{8n+6}(\mathrm{BO}(8n, \ldots.) \wedge K(\mathbf{Z}_2; 2))$$

where $\mathrm{BO}(8n, \ldots.)$ is the $(8n-1)$-connected cover of BO. It is well known that the $(8n+5)$-skeleton of $\mathrm{BO}(8n, \ldots.)$ is $S^{8n} \cup_\alpha e^{8n+4}$, where $\alpha \in \pi_3^s$ is the generator. Hence

$$\Omega_6^{\text{spin}}(K(\mathbf{Z}_2; 2)) = \pi_{8n+6}((S^{8n} \cup_\alpha e^{8n+4}) \wedge K(\mathbf{Z}_2; 2))$$

Consider the long exact sequence

$$\begin{aligned} \pi_{8n+6}(S^{8n} \wedge K(\mathbf{Z}_2; 2)) &\to \pi_{8n+6}((S^{8n} \cup_\alpha e^{8n+4}) \wedge K(\mathbf{Z}_2; 2)) \to \pi_{8n+6}(S^{8n+4} \wedge K(\mathbf{Z}_2; 2)) \\ &\to \pi_{8n+5}(S^{8n} \wedge K(\mathbf{Z}_2; 2)) \to \pi_{8n+5}((S^{8n} \cup_\alpha e^{8n+4}) \wedge K(\mathbf{Z}_2; 2)) \\ &\cong \Omega_5^{\text{spin}}(K(\mathbf{Z}_2; 2)) \end{aligned}$$

By Milgram [12] p. 77 the table $\pi_6^s(K(\mathbf{Z}_2; 2)) \cong \mathbf{Z}_2$, $\pi_5^s(K(\mathbf{Z}_2; 2)) \cong \mathbf{Z}_2$ it follows that $\Omega_6^{\text{spin}}(K(\mathbf{Z}_2; 2))$ has the order of at most 2. Since $[CP^3, s, f] \in \Omega_6^{\text{spin}}(K(\mathbf{Z}_2; 2))$ has nonzero cohomology characteristic number, where f corresponds to the generator of $H^2(CP^3; \mathbf{Z}_2)$, hence $\Omega_6^{\text{spin}}(K(\mathbf{Z}_2; 2)) \cong \mathbf{Z}_2$ and generated by $[CP^3, s, f]$. This proves the proposition.

2. CONSTRUCTION OF THE EMBEDDING

In this section we shall construct an embedding of four manifold in $\mathbf{R}^7$ with a given normal bundle by glueing the disc bundle with a suitable manifold with boundary the sphere bundle along the boundary. Our main theorem is

THEOREM 2.1. *Let M be a closed smooth four manifold, $\bar{w}_3(M) = 0$. If γ is a 3-vector bundle over M such that $\gamma \oplus \tau(M)$ trivial and the Euler class $e(\gamma)$ vanish in case $w_1(M) = 0$, then M embeds in $\mathbf{R}^7$ smoothly with the normal bundle γ if and only if there is an $x \in H^2(S(\gamma); R)$ such that its evalution at the fibre S^2 of $S(\gamma)$ is 1 and $x^3 = 0$, where $S(\gamma)$ is the sphere bundle of γ and $R = \mathbf{Z}$ or $\mathbf{Z}_2$ by $w_1(M) = 0$ or not.*

Proof. If M embeds in $\mathbf{R}^7$ with a normal bundle γ, let $B = S^7 - \mathrm{Int}D(\gamma)$ be the complement of the disc bundle of γ in the one point compactification of $\mathbf{R}^7$. A Mayer–Vietoris sequence argument gives that

$$H^2(D(\gamma); R) \oplus H^2(B; R) \to H^2(S(\gamma); R)$$

is an isomorphism. Note $e(\gamma) = 0$ since γ is a normal bundle, by Gysin exact sequence

$$0 \to H^2(M; R) \to H^2(S(\gamma); R) \xrightarrow{\phi} H^0(M; R) \to 0$$

it follows that $H^2(S(\gamma); R) \cong H^2(M; R) \oplus R$ and hence $H^2(B; R) \cong R$. Let $y \in H^2(B; R)$ be the generator and i: $S(\gamma) \to B$ be the inclusion, it is easy to check $\phi(i^*(y)) = 1 \in R$, this means that the evaluation of $i^*(y)$ at the fibre S^2 is 1. By Proposition 1.1 and 1.2 it follows that $i^*(y^3) = 0$ since $(S(\gamma), i^*(y))$ bordant to zero. This proves half of the theorem.

Conversely, if γ is indicated as the theorem, then the sphere bundle $S(\gamma)$ is a π-manifold, by Proposition 1.1 and 1.2 there exists a spin manifold B and a map $f: B \to K(R;2)$ such that $\partial B = S(\gamma)$ and $f|_B = x$. By using spin surgeries it can be assumed that $\pi_1(B) = 0$ and $f_*: \pi_2(B) \to R$ is an isomorphism. If $w_1(M) \neq 0$, consider the long exact sequence

$$H_3(S(\gamma);\mathbf{Z}) \to H_3(D(\gamma);\mathbf{Z}) \to H_3(D(\gamma), S(\gamma);\mathbf{Z}) \to H_2(S(\gamma);\mathbf{Z}) \to H_2(D(\gamma);\mathbf{Z}) \to 0$$

since $H_3(D(\gamma), S(\gamma);\mathbf{Z}) \cong \mathbf{Z}_2$ for $w_1(\gamma) \neq 0$, $H_3(D(\gamma), S(\gamma);\mathbf{Z}) \to H_3(D(\gamma), S(\gamma);\mathbf{Z}_2)$ is an isomorphism, by considering the long exact sequence above with $\mathbf{Z}_2$ coefficients it follows that the above sequence breaks into a piece of length 3 as follows

$$0 \to H_3(D(\gamma), S(\gamma);\mathbf{Z}) \to H_2(S(\gamma);\mathbf{Z}) \to H_2(D(\gamma);\mathbf{Z}) \to 0$$

and $H_3(S(\gamma);\mathbf{Z}) \to H_3(D(\gamma);\mathbf{Z})$ is an epimorphism. Moreover, one can check easily that the exact sequence above splits, hence

$$H_2(S(\gamma);\mathbf{Z}) \cong H_2(D(\gamma);\mathbf{Z}) \oplus \mathbf{Z}_2 \cong H_2(M;\mathbf{Z})) \oplus \mathbf{Z}_2$$

if $w_1(M) = 0$, similar to the above we obtain $H_2(S(\gamma);\mathbf{Z}) \cong H_2(D(\gamma);\mathbf{Z}) \oplus \mathbf{Z}$. Hence in both cases we have $H_2(S(\gamma);\mathbf{Z}) \cong H_2(D(\gamma);\mathbf{Z}) \oplus R$ and the factor R is generated by a fibre of $S(\gamma)$. As the composition $S^2 \to S(\gamma) \to K(R;2)$ is a generator, hence the inclusion i: $S(\gamma) \to B$ induces an epimorphism

$$i_*: H_2(S(\gamma);\mathbf{Z}) \to H_2(B;\mathbf{Z}) \text{ and } (i'_*, -i_*): H_2(S(\gamma);\mathbf{Z}) \to H_2(D(\gamma);\mathbf{Z}) \oplus H_2(B;\mathbf{Z})$$

is an isomorphism, where i': $S(\gamma) \to D(\gamma)$ the inclusion. A Mayer–Vietoris sequences argument shows that the smooth 7-manifold $X = D(\gamma) \cup_{S(\gamma)} B$ is 2-connected and the follows is exact:

$$H_3(S(\gamma);\mathbf{Z}) \xrightarrow{(i'_*, -i_*)} H_3(D(\gamma);\mathbf{Z}) \oplus H_3(B;\mathbf{Z}) \xrightarrow{j_1 + j_2} H_3(X;\mathbf{Z}) \to 0$$

For every element of $H_3(X;\mathbf{Z})$ can be written as $j_1(a) + j_2(b)$ for some $a \in H_3(D(\gamma);\mathbf{Z})$ and $b \in H_3(B;\mathbf{Z})$, since i'_* is onto, hence there is an $z \in H_3(S(\gamma);\mathbf{Z})$ such that $i'_*(z) = a$, note $j_1(a) + j_2(b) = j_1 i'_*(z) + j_2(b) = j_2(i_*(z) + b)$. Thus every element of $H_3(X;\mathbf{Z})$ can be represented by an embedded sphere S^3 in $\mathrm{Int}B$ since B is 1-connected. Therefore surger X by some embedded 3-spheres in $\mathrm{Int}B$ we can kill $H_3(X;\mathbf{Z})$ and obtain a homotopy sphere Σ^7, note this process does not change the tubular neighbourhood of M. Thus remove a point not lies in M we get an embedding of M in $\mathbf{R}^7$. This completes the proof.

Recall the Theorem of Boechat–Haefliger as follows:

THEOREM (Boechat–Haefliger). *Let M be an oriented closed smooth 4-manifold, $H = H^2(M;\mathbf{Z})/_{\text{torsion}}$, then M embeds in $\mathbf{R}^7$ smoothly if and only if there exists an $\omega \in H$ such that*

$$\omega^2 = sign(M), \omega x \equiv x^2 (mod\, 2)$$

Let M be an oriented closed smooth four manifold, $w_2(M) \in H^2(M;\mathbf{Z})$ the Stiefel–Whitney class, $w_2(M)$ can be lifted to an integral coefficients cohomology class $e \in H^2(M;\mathbf{Z})$. Let $e_1, \ldots, e_n \in H^2(M;\mathbf{Z})$ be a series of primitive elements which project to H form a basis of it, let T be the torsion subgroup of $H^2(M;\mathbf{Z})$, then we have a direct sum decomposition $H^2(M;\mathbf{Z}) \cong \langle e_1, \ldots\ldots, e_n \rangle \oplus T$ where $\langle e_1, \ldots\ldots, e_n \rangle$ denote the span of $e_1, \ldots\ldots, e_n$. Denote by $\bar{\omega} = \Sigma_{1 \leqslant i \leqslant n} a_i e_i$ the element corresponding to $\omega \in H$ as the above theorem, $e = \Sigma_{1 \leqslant i \leqslant n} x_i e_i + y$, $y \in T$. Since $ex \equiv w_2(M)x \equiv v_2(M)x \equiv x^2 (\mathrm{mod}\, 2)$ for all $x \in H^2(M;\mathbf{Z})$, where $v_2(M)$ the Wu class of M, hence $a_i = x_i (\mathrm{mod}\, 2)$ for $1 \leqslant i < n$. Let

$\mathbf{z} = \Sigma_{1 \leqslant i \leqslant n} a_i e_i + y \in H^2(M; \mathbf{Z})$, $\mathbf{z}$ (mod 2) $= w_2(M)$, and $z^2 = \bar{\omega}^2 = \mathrm{sign}(M)$. This shows that the existence of $\omega \in H$ in the above theorem is equivalent to the class $\mathbf{z} \in H^2(M; \mathbf{Z})$ such that $\mathbf{z}$ (mod 2) $= w_2(M)$ and $\mathbf{z}^2 = \mathrm{sign} M$.

Proof of Boechat–Haefliger's Theorem. If M embeds in $\mathbf{R}^7$ smoothly, by Theorem 2.1 there exists an $x \in H^2(S(\gamma); \mathbf{Z})$ such that the evalution of x at the fibre S^2 is 1 and $x^3 = 0$. Consider the Gysin exact sequence

$$0 \to H^2(M; Z) \to H^2(S(\gamma); \mathbf{Z}) \xrightarrow{\phi} H^0(M; \mathbf{Z}) \to 0$$

Note $\phi(x) = 1$, $\phi(x^3) = 0$. By [8] there are two elements $\alpha \in H^4(M; \mathbf{Z})$ and $\beta \in H^2(M; \mathbf{Z})$ such that $x^2 = \pi^*(\alpha) + x \cup \pi^*(\beta)$ and $\beta \bmod 2 = w_2(\gamma) = v_2(M)$, $4\alpha + \beta^2 = p_1(\gamma) = -3\mathrm{sign} M$. Hence $0 = \phi(x^3) = \alpha + \beta^2 = 3/4(\beta^2 - \mathrm{sign} M)$, i.e., $\beta^2 = \mathrm{sign} M$ and $\beta(\bmod 2) = v_2(M)$ which implies $\beta x = x^2 (\bmod 2)$, this proves one direction.

Conversely, by the argument below the theorem we have an $\omega \in H^2(M; \mathbf{Z})$ satisfies $\omega^2 = \mathrm{sign} M$, $\omega(\bmod 2) = w_2(M)$. Up to homotopy ω gives a map $\omega: M \to CP^2$. Let ξ be the normal bundle of CP^2 in $\mathbf{R}^7$, $\gamma = \omega^* \xi$ be the pullback bundle of ξ, then $p_1(\gamma) = -3\omega^2 = -3\mathrm{sign} M$, $w_2(\gamma) = \omega(\bmod 2) = w_2(M)$. By Dold–Whitney [5] it follows that $\gamma \oplus \tau(M)$ is a trivial bundle. Note the Euler class $e(\gamma) = 0$. Let f: $S(\gamma) \to S(\xi)$ be the sphere bundle map over ω, by Theorem 2.1 there is an $x \in H^2(S(\xi); \mathbf{Z})$ such that its evalution at the fibre is 1 and $x^3 = 0$. Hence the evalution of $f^*(x)$ at the fibre of $S(\gamma)$ is also 1 and $(f^*(x))^3 = 0$. By using Theorem 2.1 again we can embed M in $\mathbf{R}^7$ with normal bundle γ. This completes the proof.

Let γ be as Theorem 2.1, $w_1(M) \neq 0$, $x \in H^2(S(\gamma); \mathbf{Z}_2)$ evolutes at the fibre 1, we remark that the value $x^3 \in \mathbf{Z}_2$ does not depend on the choice of the class with evalution 1 at the fibre. This follows directly from Massey [11] Theorem III. We summary it as

PROPOSITION 2.2. *Let M be a closed smooth nonorientable four manifold, γ be a 3-vector bundle over M, $w_3(\gamma) = 0$, $w_2(\gamma) = v_2(M)$. If x and x' are two elements of $H^2(S(\gamma); \mathbf{Z}_2)$ both evalute 1 at the fibre of $S(\gamma)$, then $x^3 = x'^3$.*

Proof of main Theorem (A). By Donaldson Theorem [6] and the above Boechat–Haefliger Theorem the orientable case follows. Hence we consider only the nonorientable case. It is well known that $\bar{w}_3(M) = 0$ is a necessary condition for M embeds in $\mathbf{R}^7$, now we prove the other direction. Let γ be a 3-vector bundle over M such that $\gamma \oplus \tau(M)$ trivial, this is possible as M can be immersed in $\mathbf{R}^7$. Note $H^4(M; \mathbf{Z}) = \mathbf{Z}_2$ and the mod 2 reduction $H^4(M; \mathbf{Z}) \to H^4(M; \mathbf{Z}_2)$ is an isomorphism. By Dold–Whitney [5] it follows that the stable orientable vector bundle over M is classified by the Stiefel–Whitney classes w_2 and w_4. Let α: $S^4 \to BO(3)$ be a generator of $\pi_4(BO(3)) \cong \mathbf{Z}$, note the sphere bundle of α is just CP^3. Let $\gamma: M \to BO(3)$ be the classifying map of γ, $p: M \to M \vee S^4$ be the pinch map, it is easy to check that the composition $g = (\gamma \vee \alpha) \circ p$: $M \to BO(3)$ gives also a normal bundle of an immersion of M in $\mathbf{R}^7$. Denote by $S(\zeta)$ the sphere bundle of g and $S(\xi)$ the sphere bundle of $(\gamma \vee \alpha)$, consider the following commutative diagram of Gysin exact sequences:

$$\begin{array}{ccccccc} 0 \to & H^6(S(\xi); \mathbf{Z}_2) & \xrightarrow{\phi} & H^4(M \vee S^4; \mathbf{Z}_2) & \to 0 \\ & \downarrow{\scriptstyle \bar{p}^*} & & \downarrow{\scriptstyle p^*} & \\ 0 \to & H^6(S(\zeta); \mathbf{Z}_2) & \xrightarrow{\phi} & H^4(M; \mathbf{Z}_2) & \to 0 \end{array}$$

Note $H^6(S(\xi);\mathbf{Z}_2) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ with two generators $\mathbf{z}_1$ and $\mathbf{z}_2$ satisfying $\phi(\mathbf{z}_1) = \{\zeta_M\}$, $\phi(\mathbf{z}_2) = \{\zeta_{S^4}\}$ where $\{\zeta_-\}$ denotes the mod 2 cohomology fundamental class. If $a \in H^2(S(\xi);\mathbf{Z}_2)$ is a generator with $\phi(a) = 1$, i.e., its evalution at the fibre is 1, note the restriction of $S(\xi)$ over M and S^4 gives the sphere bundle $S(\gamma)$ and CP^3 resp, by Proposition 2.2 we get $a^3 = \lambda \mathbf{z}_1 + \mathbf{z}_2$, where $\lambda \in \mathbf{Z}_2$ is the cube of the restriction of a at $S(\gamma)$. By the diagram above we have

$$\phi((\bar{p}^* a)^3) = p^* \phi(a^3) = p^*(\lambda\{\zeta_M\} + \{\zeta_{S^4}\}) = (\lambda + 1)\{\zeta_M\}$$

Therefore the value $(\bar{p}^* a)^3$ change by 1 and thus there exists an bundle satisfies the condition of Theorem 2.1. This completes the proof.

Proof of Corollary 1 *in the smooth case.* Let g: $M \to \mathbf{R}^7$ be an immersion, γ_g its normal bundle and $a \in H^2(S(\gamma_g);\mathbf{Z}_2)$ evaluates at the fibre 1, if $a^3 = 1$, by Theorem 2.1 and Proposition 2.2 γ_g can't become the normal bundle of any embedding of M in $\mathbf{R}^7$. If $a^3 = 0$, by the proof of **main** Theorem (A) we can obtain another immersion f: $M \to \mathbf{R}^7$ with normal bundle γ_f by changing γ_g at the top dimension cell, and such that for an $a_1 \in H^2(S(\gamma_f);\mathbf{Z}_2)$ evalute 1 at fibre, then $a_1^3 = 1$. This completes the proof.

3. THE PROOF IN THE TOPOLOGICAL CATEGORY

The proof in the topological case follows the same idea and in the following we discuss the necessary changes.

LEMMA 3.1. *Let $S(\gamma)$ be a S^2-bundle over a closed topological 4-manifold with structure group $SO(3)$, $w_3(\gamma) = 0$, then there is an element $a \in H^2(S(\gamma);\mathbf{Z}_2)$ whose evalution at the fibre* 1 *and satisfying*

$$\langle KS(S(\gamma)) \cup a, [S(\gamma)]_2 \rangle = \langle KS(M), [M]_2 \rangle$$

where $KS(\gamma)$ is the Kirby–Siebenmann obstruction and $[\ \ ]_2$ the mod 2 fundamental class.

Proof. As we denote in Section 2, ϕ: $H^{2+q}(S(\gamma);\mathbf{Z}_2) \to H^q(M;\mathbf{Z}_2)$ is the homomorphism in the Gysin exact sequence, $w_3(\gamma) = 0$ implies that there is a class $a \in H^2(S(\gamma);\mathbf{Z}_2)$ so that $\phi(a) = 1$. Let p: $S(\gamma) \to M$ denote the projection of the bundle, the tangent bundle of $S(\gamma)$ satisfies

$$\tau(S(\gamma)) \oplus \varepsilon \cong p^*(\tau M) \oplus p^*\gamma$$

and hence $KS(S(\gamma)) = p^*(KS(M)) + p^*(KS(\gamma)) = p^*(KS(M))$.

Note $\phi: H^6(S(\gamma);\mathbf{Z}_2) \to H^4(M;\mathbf{Z}_2) \cong \mathbf{Z}_2$ is an isomorphism and thus one has

$$\phi(KS(S(\gamma)) \cup a) = \phi(p^*(KS(M) \cup a) = KS(M) \cup \phi(a) = KS(M)$$

this completes the proof.

Comparing with the smooth case to see what need to be changed, in Section 1 one has to compute the corresponding bordism group. The results are

PROPOSITION 3.2(i). $\Omega_6^{Topspin}(K(\mathbf{Z};2)) \cong \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_2$

(ii). $\Omega_6^{Topspin}(K(\mathbf{Z}_2;2)) \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$

and the following are monomorphisms

(i). $[M, s, f] \mapsto (\langle f^*(x^3), [M]\rangle, \langle p_1(M) \cup f^*(x), [M]\rangle, \langle KS(M) \cup f^*(x), [M]_2\rangle)$

(ii). $[M, s, f] \mapsto (\langle f^*(x^3), [M]_2\rangle, \langle KS(M) \cup f^*(x), [M]_2\rangle)$

Proof. By Atiyah–Hirzebruch sequence it follows that $\Omega_6^{\text{Topspin}}(K(\mathbf{Z}; 2)) \cong \mathbf{Z} \oplus \mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_2$, and the torsion free part is detected by the first two characteristic numbers states in (i). Now we construct a nonzero torsion element. Let CH be the Chern manifold in [7], H the Hopf bundle over CP^2, $f: CH \to CP^2$ denote an orientation preserving homotopy equivalence, $S(H \oplus \varepsilon)$ the sphere bundle of $H \oplus \varepsilon$, then $S(H \oplus \varepsilon)$ and $S(f^*H \oplus \varepsilon)$ are spin manifolds. By the same reason as Lemma 3.1 there is a class $a \in H^2(S(H \oplus \varepsilon); Z)$ whose reduction mod 2 satisfies Lemma 3.1, consider the bordism class in $\Omega_6^{\text{Topspin}}(K(\mathbf{Z}; 2))$, $[S(H \oplus \varepsilon), s, a] - [S(f^*H \oplus \varepsilon), \bar{f}^*s, \bar{f}^*a]$, where $\bar{f}$ is the bundle map over f and s an arbitrary spin structure on $S(H \oplus \varepsilon)$. It is obvious that the first two characteristic numbers of the above bordism class are zero, by Lemma 3.1 the third characteristic number is $KS(CH) \in \mathbf{Z}_2$, which is nonzero by [7], this proves (i).

To prove (ii), first note that the nonzero E_2-term of the Atiyah–Hirzebruch sequence of $\Omega_6^{\text{Topspin}}(K(\mathbf{Z}_2; 2))$ as the following, $E_2^{2,4} \cong \mathbf{Z}_2$, $E_2^{4,2} \cong \mathbf{Z}_2$, $E_2^{6,0} \cong \mathbf{Z}_2$. For smooth case, the E_2-term of corresponding bordism group are same as that and in the range $(*, q)$, $q < 4$, the differentials are also same, hence by Proposition 1.2, one knows that $E_2^{4,2}$ must be hit by the differential d_2 or d_3, thus $E_2^{4,2}$ doesn't survive in $E_\infty^{4,2}$. This argument shows that the order of group $\Omega_6^{\text{Topsipn}}(K(\mathbf{Z}_2; 2))$ is less than or equal to 4. On the other hand, by check the two invariants in Proposition 3.2(ii) for the bordism class (CP^3, s_1, x) and $[S(H \oplus \varepsilon), s, a] - [S(\bar{f}^*H \oplus \varepsilon), \bar{f}^*s, \bar{f}^*a]$, where s_1 is the unique spin structure on CP^3 and $x \in H^2(CP^3; \mathbf{Z}_2)$ the generator, the conclusion (ii) follows immediately.

The arguments of Section 2 work identically in the topological category, one only has to see that $KS(M) = 0$ is a necessary condition for locally flat embedding M in R^7. In fact, since a locally flat embedded 4-manifold M has a normal bundle $\gamma: M \to \text{BTOP}(3)$, which can be lifted to BDiff(3) since the homotopy group $\pi_q(\text{TOP}(3)/_{\text{Diff}(3)}) = 0$ for $q < 6$ [10], this implies $KS(M) = KS(\gamma) = 0$. This completes the proof of the main Theorem and Corollary 1.

Proof of Corollary 2. In case (i), [2] has stated that the condition about quadratic form in the Main Theorem B(ii) is fulfilled. In case (ii), by [13], there is always a class $\bar{\omega} \in H$ so that $\bar{\omega} = n/4$, where $n = \text{rank } H_2(M; \mathbf{Q}) = \text{sign} M$. Let $\omega = 2\,\bar{\omega}$, which satisfies the requirement in the Theorem B(ii).

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